Understanding the Hebrew Calendar

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1 Introduction

Calendar design is generally based on three astronomical cycles: the rotation of the earth about its axis, the revolution of the earth about the sun, and that of the moon about the earth. A calendar may be classified as *lunar*, *solar*, or *lunisolar* according to the combination of these phenomena on which it depends.

Solar calendars are designed to track the sun. This is achieved by arranging for the average length of a year, measured in days, to be as close as possible to the mean period of the earth's revolution, known as the *tropical year*, which has been calculated to be 365.2422 days.[1] The civil calendar instituted during the reign of Julius Caesar approximated this objective with a four-year cycle consisting of three 365-day years and one leap year of 366 days, yielding an average year length of 365.25 days. This was an overestimate by more than 11 minutes, resulting in an accumulated *solar drift* of some 12 days over the next 16 centuries. In 1582, Pope Gregory II replaced the Julian calendar with a refinement, eliminating three leap years every 400 years (those that are divisible by 100 but not 400). This results in an average year length of $365+97/400=365.2425$ days, a better approximation to the tropical year. The Gregorian calendar was gradually accepted universally (adopted by the British Empire in 1752) and remains in dominant use today.

Lunar calendars are based on the cycle of phases of the moon, which is delimited by successive *lunar conjunctions*. A conjunction occurs when the moon lies between the earth and the sun, i.e., the earth lies on the dark side of the moon. But since the plane of the earth's orbit, known as the *ecliptic*, is distinct from that of the moon's (forming an angle of slightly more than 5 degrees), it is unusual for the moon to cross the line that connects the earth and the sun. More precisely, a conjunction occurs when the moon lies on the plane perpendicular to the ecliptic that contains that line. As illustrated in Figure 1, the period between successive conjunctions, known as the *synodic month*, is distinct from the period of revolution of the moon. As viewed from above the earth's north pole, both revolutions are counterclockwise. Note that once the moon has completed a full revolution from a point of conjunction, it must traverse an additional arc before reaching the next conjunction. Thus, a synodic month is somewhat longer than the period of revolution. The eccentricity of the earth's orbit results in significant variation in its angular velocity with respect to the sun and concommitant variation in the length of a synodic month, but the mean synodic month has been estimated with considerable accuracy to be 29.53059 days. The period of lunar revolution, on the other hand, is just over 27 days.

A lunar calendar divides the year into months of lengths that are selected to ensure a close correlation between calendar and synodic months, thereby minimizing *lunar drift*. For example, the civil Islamic calendar has a year consisting of 12 months, alternating between 30 and 29 days in length. To account for the resulting slight underestimate of the mean synodic month, the final month of the year is extended by a

[1] As a simplification at the expense of some degree of accuracy, we shall ignore variations such as the tidal slowing of the earth's rotation and treat this value as a constant.

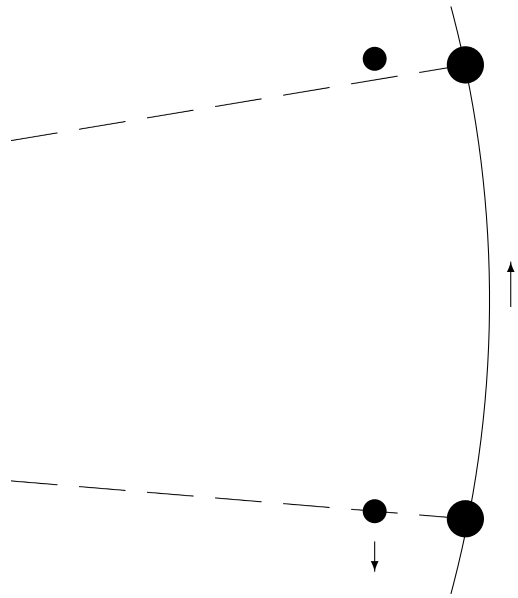

Figure 1: Lunar cycle

day in 11 of every 30 years to reduce drift. The average year length, therefore, is $30 \cdot 6 + 29 \cdot 6 + 11/30 \approx 354.367$ days, and the average month length is about 29.53056 days. Since the former is nearly 11 days less than the tropical year, this calendar exhibits rapid solar drift. If Ramadan, for example, occurs in the summer of a given year, then over the next 12 years it will shift to the winter.

The Hebrew calendar is lunisolar, meaning that it tracks both the sun and the moon. The correlation with the lunar cycle is dictated by the biblical tradition of sanctifying the new moon on the first day of each month. On the other hand, the traditional agricultural components of various Jewish holidays require alignment with the tropical year: Pesach (Passover) is also known as *chag ha'aviv*, the festival of the spring, and Sukkot is tied to the fall harvest. This is enforced by varying the number of months in a year, arranging that the average number be close to the ratio of the tropical year to the synodic month,

$$\frac{365.2422}{29.53059} \approx 12.36827.$$

Thus, at roughly three-year intervals, an extra month is inserted into the 12-month *common year* to produce a 13-month *leap year*.

In ancient Israel, this dual alignment was achieved dynamically through empirical observation. *Rosh Chodesh*, literally the "head" of the month, was proclaimed whenever the first sighting of the crescent moon was reported by witnesses to the *Sanhedrin* (supreme court) of Jerusalem, whether this occurred on the 29$^{\text{th}}$ or the 30$^{\text{th}}$ day after the first of the preceding month. The renowned 12$^{\text{th}}$ century philosopher and astronomer Rabbi Moshe ben Maimon, known by the Greek name Maimonides as well as the acronym Rambam, is a primary source for this history. According to Rambam's *Sanctification of the New Month* [9] (1178 C.E.), the proclamation was initially spread though the diaspora by lighting fires:

> Originally it was customary, when the court sanctified the new moon, to light fire signals on the tops of mountains, so that those who lived at a distance might learn of it. But when

> the Cutheans began to cause trouble by kindling fire signals in a mischievous way, in order to mislead the people, a law was enacted whereby messengers were sent out to inform the public [9, Chapter III].

The Sanhedrin was also responsible for determining leap years, based primarily on the anticipated proximity of the spring season at the end of the month of Adar, the final month of winter. If it was judged that the barley season would not arrive in time for Pesach, which begins on the 15$^{\text{th}}$ of the following month of Nisan, then Nisan would be delayed by the insertion of a second Adar.[2] This decision was also announced beyond Jerusalem by messengers.

Eventually, of course, these practices were replaced by a static calendar based on calculation rather than observation. Rambam continues:

> … in times when no sanhedrin existed, this declaration was based on calculations such as we are using today … Since when did all of Israel begin to employ these methods of calculation? Since the time of the last sages of the Gemara; that was the time when Palestine was destroyed and no regularly established court was left [9, Chapter V].

Today, the origin of the fixed calendar is generally associated with Hillel ben Yehudah, known as Hillel II, who has been credited with its institution in the Hebrew year 4119 (358–359 C.E.), when he was head of the Sanhedrin. Hillel was not the last such leader, but there were a variety of political reasons for abandoning the old practices prior to the disbanding of the court, not the least of which was the prohibition of the announcement of Rosh Chodesh by the emperor Constantius II. Some of the details of Hillel's calendar are in doubt, but it is generally believed to have subsequently evolved, perhaps through the 9$^{\text{th}}$ century, before reaching its final form. We shall touch on various aspects of this history, but our main objective is to elucidate the calculations and conventions that underly the Hebrew calendar in its present state.[3] We shall also describe a computer program that generates the calendar, converts between Hebrew and Gregorian dates, and serves as a basis for the formal analysis and verification of some critical properties of the calendar.

2 Computation of the Moladot

A first step in the design of a lunisolar calendar is the determination of the leap years. The Hebrew calendar employs a scheme that was used by the Greeks and Babylonians long before Hillel, known as the *Metonic cycle* (after the Athenian astronomer Meton of the 5$^{\text{th}}$ century B.C.E.), based on the observation that the ratio of the tropical year to the synodic month (noted above) is closely approximated by the ratio

$$\frac{235}{19} \approx 12.36842.$$

This gives rise to a 19-year cycle consisting of 12 common years and 7 leap years, with a total of $12 \cdot 12 + 7 \cdot 13 = 235$ months and an average of $235/19$ months/year. Thus, by convention, a Hebrew year is a leap year if its remainder upon division by 19 is one of the following: 0, 3, 6, 8, 11, 14, and 17. A leap year, therefore, occurs every two or three years. The 12 months of a common year are Tishri,

[2] During the early period of Christianity, when the date of Easter depended on the Hebrew calendar, there was also pressure from the Church to ensure that the full moon of Nisan did not precede the vernal equinox.[1, 2]

[3] In particular, this exercise was largely motivated by the author's experience of going through life wondering why the months of Cheshvan and Kislev have 29 days in some years and 30 in others, and what might be the basis for this determination.

Cheshvan, Kislev, Tevat, Shevat, Adar, Nisan, Iyar, Sivan, Tammuz, Av, and Elul.[4] The extra month of a leap year is designated as an additional month of Adar, named Adar I and inserted before the common month of Adar, which becomes Adar II.[5] Each month is assigned a length of either 29 or 30 days as specified in Section 3.

The next step is an estimate of the time of the lunar conjunction that marks the beginning of each month. This estimate is called a *molad*, from the root of the Hebrew word for "birth", with plural *moladot*. The moladot are commonly specified by two parameters: (1) an estimate of the mean synodic month, which we shall represent as $\mathscr{L}$, and (2) the molad of a single representative month, to be used as a point of reference. Every other molad may then be computed by shifting the reference molad by an appropriate multiple of $\mathscr{L}$.

The conjunction, as we have noted, occurs when the earth is on the dark side of the moon, and in fact, neither the waxing nor waning crescent moon is generally visible within 24 hours of this event. How, then, were conjunctions identified by ancient astronomers? In the rare instance when the moon lies precisely between earth and sun, the conjunction is observable as a solar eclipse. An approximation of $\mathscr{L}$ may be derived by dividing the time elapsed between two recorded eclipses, perhaps several centuries apart, by the number of intervening months. This observation was exploited by Babylonian and Greek astronomers in the derivation of a remarkably accurate estimate. Traditionally, for the purpose of representing such quantities, an hour is divided into 1080 *parts*, where a part of an hour is therefore $1/18$ of a minute or 3 $1/3$ seconds. The value of $\mathscr{L}$ recorded by Ptolemy in his *Almagest* in the 2nd century C.E., which apparently had been established several centuries earlier, is 29 days, 12 hours, and 793 parts. This value remains the basis of today's Hebrew calendar.

The times of the moladot are traditionally based on an imprecisely specified meridian in the vicinity of Jerusalem. Following Rambam, we shall adopt the convention that the day begins at 6 PM and the time of day is measured accordingly. For example, 4 hours, 0 parts is 10 PM and 16 hours, 540 parts is 10:30 AM. By Hebrew convention, the days of the week are numbered, with the first day beginning at 6 PM on Saturday, near the end of Shabbat. We shall adhere to the usual abuse of notation in the use English names to refer to Hebrew days, e.g., "Monday" is understood to refer to the second day of the week, i.e., the 24-hour period that begins on 6 PM on Sunday.

According to Ajdler [1], the reference molad identified by Hillel was the moment of the solar eclipse that occurred on the afternoon of March 15, 359 C.E. (under the Julian calendar), the day of the inception of the fixed calendar, which thus became the molad of Nisan, 4119. One deficiency of Hillel's calendar stemmed from the convention in that era of dividing an hour into coarser units of 4 minutes rather than 1080 parts. Consequently, Hillel's approximation of $\mathscr{L}$ was 29 days, 12 hours, and 44 minutes. Since 44 minutes equates to 792 parts, this is one part short of the more accurate interval reported by Ptolemy. By the year 4537, this inaccuracy resulted in an accumulated drift of nearly 5 hours. Perhaps for that reason, the molad of Tishri of that year, which would be on a Tuesday at 18 hours, 1008 parts according to Hillel's calculations, was officially adjusted to the start of the next day, at 0 hours, 0 parts [1].

It is likely that the more accurate ancient value of $\mathscr{L}$ was adopted by the rabbis in the 9th century, when the *Almagest* was translated into Arabic. But another critical deficiency of Hillel's calendar was that the reference molad was based on a single data point. A more prudent approach would be to examine a large set of observed conjunctions spanning a long period and select a reference molad that provides the best linear fit of the available data. Since the variation of the earth's angular velocity results in

[4] Note that while the Torah identifies Nisan as the first month of the year, the civil ordering listed here serves our purpose, since Tishri 1 is the day on which the year numbering changes.

[5] A consequence of this ordering is that in a leap year, Purim is celebrated on the 14th day of Adar II.

deviations up to 13 hours from the mean, it is unlikely that an arbitrarily chosen single conjunction will yield anything close to an optimal solution.

The final molodot that are in use today are based on the molad of Tishri in the year 2, which is taken to have occurred at precisely 14 hours on a Friday. Owing to the secrecy with which calendrical calculations were treated by the medieval sages, history provides no clear record of the origin of this reference molad. There is evidence, however, in support of speculation[1, 12] that it too was based on the *Almagest*. It appears to have been derived from a value found in Ptolemy's tables, namely, a mean conjunction corresponding to Nisan in the year 3014, computed to have occurred in Alexandria on a Saturday at 11 hours, 770 parts. Since the difference in longitude between Alexandria and Jerusalem of $5°30'$ results in a time difference of

$$\frac{5.5}{360} \cdot 24 \text{ hours} = 22 \text{ minutes} = 396 \text{ parts},$$

this translates to 12 hours, 86 parts in Jerusalem. The molad of Tishri of the year 2 is computed, according to the Metonic cycle, to have occurred 37,260 months earlier. Subtracting $37{,}260 \cdot \mathscr{L}$ from this computed molad of Tishri 3014 yields 13 hours, 626 parts on a Friday, which was apparently rounded up to 14 hours.

Note that according to tradition, this was 8 AM on the sixth day of creation, which is assumed to have taken place during the final week of the year 1. Another reason for the mystery surrounding this event, which is known as *Molad Adam*, is the associated mythology. In lieu of hard scientific records, there is a popular belief that the details of the calendar, including Molad Adam, are *Torah l'Moshe mi'Sinai*, part of the oral Torah conveyed to Moses on Mt. Sinai. According to Talmudic commentary, apparently based on the long-held erroneous belief that the moon becomes visible 6 hours after the conjunction, the newborn Adam witnessed the first crescent moon at 2 PM on that day [3].

Whatever the origin of the modern moladot, the underlying analysis was sufficiently sophisticated to produce a reasonably accurate solution to the statistical problem. Evidence supporting this conclusion is provided by the the scatter plot of Figure 2, which is due to Bromberg [3]. The graph covers a period of 124,000 months, approximately 10,000 years, beginning in the Hebrew year 1. For each displayed data point, the horizontal coordinate represents a given month and the vertical coordinate is the difference D, measured in minutes, between the calculated molad and the actual lunar conjunction for that month. We would expect that if the molodot are optimally computed, then these points should lie in a strip centered on the horizontal axis. If we confine our attention to a period of several centuries around the time of Ptolemy (50,000 months $\approx$ 4043 years, which corresponds to 282 C.E.), then the graph fits this description with D ranging roughly between ± 13 hours, but over the entire domain, the strip appears to be parabolic rather than constant, with the range shifting in the present era to $-12 \text{ hours} \leq D \leq 14 \text{ hours}$. The reason for this is that our expectation is based on the assumption that the constant molad interval $\mathscr{L}$ accurately represents the synodic month S over this period, but in fact, S has been decreasing linearly. 2000 years ago, S was within a few microseconds of $\mathscr{L}$; by today it has decreased by more than a half second. Since the details of this analysis are irrelevant to an understanding of the principles underlying the design of the calendar, the next paragraph (which departs from Bromberg's analysis) may be ignored by the squeamish reader without loss of continuity.

If we assume that $S(t_0) = \mathscr{L}$ at some time t_0 (measured in months) and that the rate of linear decrease of $S(t)$ is m, then

$$S(t) = \mathscr{L} - m(t - t_0).$$

Thus, the slope of the curve at t months, i.e., the increase of D over a period of 1 month, is $\mathscr{L} - S(t) =$

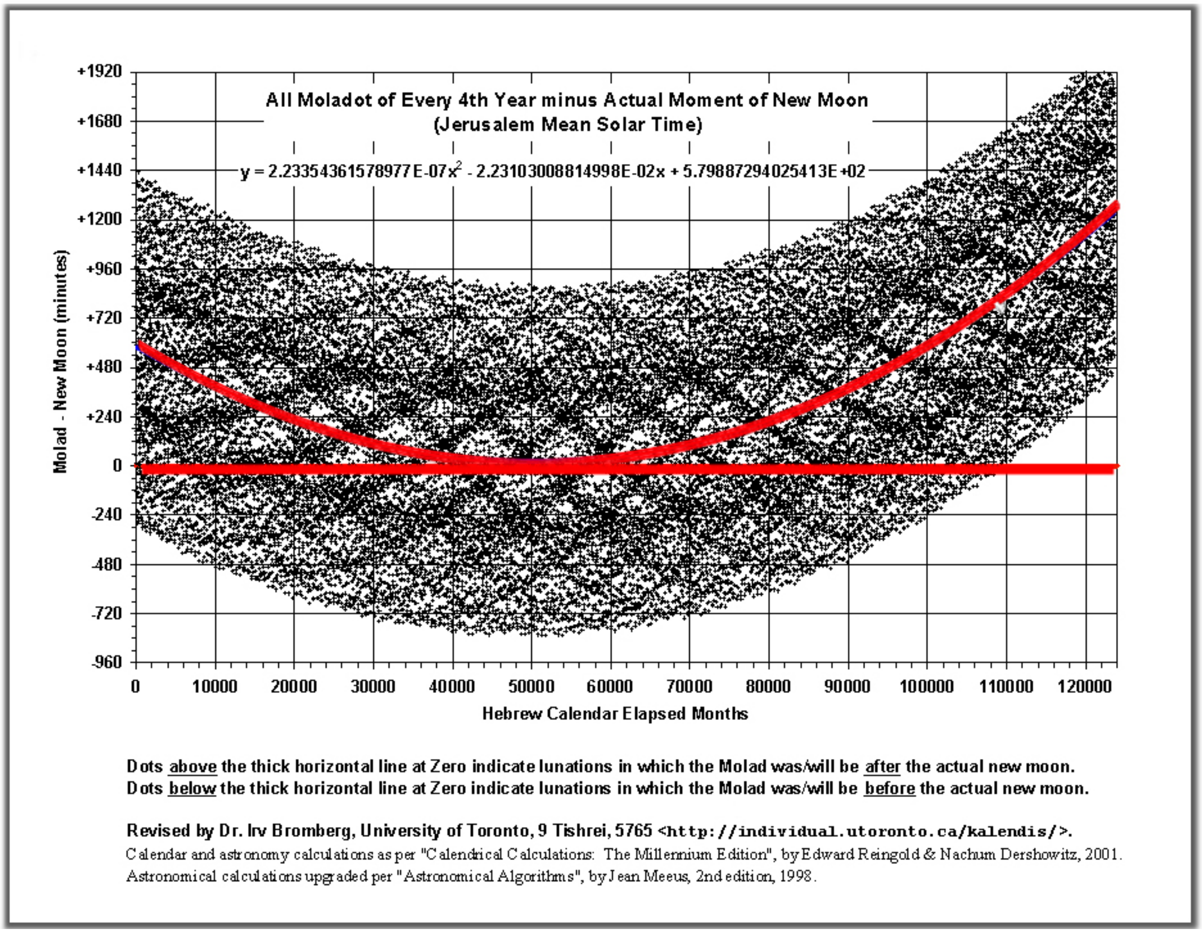

Figure 2: Deviation of molodot from astronomically computed conjunctions

$m(t-t_0)$, and it follows that

$$D(t) = \int_{t_0}^{t} m(t-t_0)dt = \frac{1}{2}m(t-t_0)^2.$$

This is consistent with the parabolic shape of the graph. According to Bromberg's computed equation for the curve, the coefficient $m/2$ is approximately 2.23×10^{-7} minutes/month, and we conclude that S is decreasing at a rate of

$$m \approx 2 \times 2.23 \times 10^{-7} \text{ minutes/month} \approx 27 \text{ microseconds/month}.$$

In modern analysis of the calendar, Molad Adam has been replaced as a point of reference by the (imaginary) molad of Tishri of year 1, computed by subtracting $12 \cdot \mathscr{L}$ from Molad Adam, resulting in 5 hours, 204 parts on the second day of the week. This is known as *Molad Beharad*, a transliteration of the Hebrew בהרד, which is derived from the conventional assignment of numerical values to letters (ב = 2, ה = 5, ר = 200, ד = 4). We shall find that our analysis is facilitated by introducing an *absolute calendar*, which is nothing more than a sequential numbering of days. Since Beharad falls on the second day of the week, it is natural to set the first day of this calendar to the preceding day, so that the day of the week can be readily derived from an absolute date as the remainder upon by division by 7.

In the following sections, we shall represent a duration of d days, h hours, and p parts as $(d:h:p)$, e.g., $\mathscr{L} = (29:12:793)$. We use the same notation for a moment in time, using the absolute date for d. Thus, Molad Beharad is $\mathscr{B} = (2:5:204)$. The addition of these objects and multiplication by an integer are defined in the natural way. Using these values of $\mathscr{B}$ and $\mathscr{L}$ together with the number of months in each year as determined by the Metonic cycle, we may compute the absolute date and time of any molad, e.g., Molad Adam is

$$\mathscr{B} + 12 \cdot \mathscr{L} = (356:14:0)$$

and the molad of Tishri, 5782 is

$$\mathscr{B} + 71489 \cdot \mathscr{L} = (2111469:5:497).$$

A time of day may be represented using the same notation, omitting the date component, i.e., as $(h:p)$. Times of day are compared in the natural way. Thus, $(h_1:p_1) \leq (h_2:p_2)$ means that $h_1 + p_1/1080 \leq h_2 + p_2/1080$.

3 Determination of Rosh Hashanah

In view of the value of the molad interval $\mathscr{L}$, the month lengths are tentatively set to alternate between 30 and 29 days, beginning with the 30-day month of Tishri. Since the resulting average length is a slight underestimate of $\mathscr{L}$, the extra month of a leap year, Adar I, is naturally assigned 30 days. This scheme would result in common and leap years of lengths $6 \times 29 + 6 \times 30 = 354$ and and $354 + 30 = 384$, respectively.

Rosh Hashanah, the new year, is traditionally celebrated during the first two days of Tishri but we shall use the term to refer to the single day Tishri 1. Unless otherwise specified, the term *molad* may be assumed to refer to the molad of Tishri. A naive objective is to arrange for the molad of each year to occur on Rosh Hashanah. This is not generally possible, however, with year lengths as specified above. For example, the molad of a common year is separated from that of the following year by an interval of $12 \cdot \mathscr{L} = (354:8:817)$. Consequently, depending on the time of day of the first of these moladot, the absolute dates on which they occur may differ by either 354 or 355 days. Similarly, since $13 \cdot \mathscr{L} = (383:21:589)$, the absolute dates of the moladot that begin and end a leap year may differ by either 383 or 384. All of these possible year lengths may be accommodated by allowing the lengths of two specific months, Cheshvan and Kislev, to vary between 29 and 30, as displayed in Figure 3. Thus, if the length of a common year is computed (from the absolute dates of the bounding moladot) to be 355, then the lengths of Cheshvan and Kislev should both be set to 30. Similarly, if the length of a leap year is found to be 383, then both months are assigned 29 days. The first case is called a *complete* common year, the second is a *defective* leap year, and a year of length 354 or 384, in which Cheshvan has 29 days and Kislev has 30, is said to be *regular*.

Of course, this scheme is far too simple to serve as a rabbinical solution. As the calendar evolved between the fourth and ninth centuries, a variety of constraints were imposed, resulting in the possible delay of Rosh Hashanah by up to two days beyond the date of the molad. This led to additional variation in the length of a year, as determined by the absolute dates of Rosh Hashanah of that year and the next. We shall demonstrate, however, that the prescribed Rosh Hashanah postponements, or *dechiyot*, effectively ensure that the resulting length of a common year is always 353, 354, or 355, and that of a leap year is 383, 384, or 385. Consequently, the month lengths listed in Figure 3 remain in force, but with the new understanding that either a common year or a leap year may be defective (each of Cheshvan and Kislev has 29 days), regular (Cheshvan has 29 and Kislev has 30), or complete (each has 30).

Month	Days
Tishri	30
Cheshvan	29 or 30
Kislev	30 or 29
Tevat	29
Shevat	30
Adar I (leap year only)	30
Adar (Adar II on leap year)	29
Nisan	30
Iyar	29
Sivan	30
Tammuz	29
Av	30
Elul	29

Figure 3: Months of the year

The four dechiyot are traditionally listed in the following order:

First Dechiyah: *If the molad of Tishri occurs at or after noon, then Rosh Hashanah is postponed to the next day.*

The rationale for this postponement is far from clear. A common explanation is that the rule is intended to ensure that the new moon is visible in the sky during the first evening of Rosh Hashanah. This is consistent with the erroneous belief mentioned above in connection with Molad Adam, but is unrealistic since the moon rarely appears within 24 hours of the molad of any month. An alternative reason is suggested by Landau [7]: this delay guarantees that the molad of every month always occurs before the end of the first day of the month. (Demonstration of this property requires some computation; see Section 5.) Another novel explanation of the delay is offered by Fiedler [5], who argues that its purpose is to prevent the visibility of the waning moon of Elul on Rosh Hashanah.

Whatever the motivation for this dechiyah, its simplest implementation is to replace the molad with the moment that occurs 6 hours later, which we shall call the *delayed molad*, thereby effectively bypassing the 1st dechiyah. That is, we begin with the assumption that Rosh Hashanah is tentatively scheduled for the day of the delayed molad and replace the usual statements of the remaining dechiyot with reformulations expressed in terms of the delayed molad.[6]

Second Dechiyah: *If the delayed molad occurs on a Wednesday, Friday, or Sunday, then Rosh Hashanah is postponed to the next day.*

While a variety of conjectures have been offered regarding the origins of the prohibitions of these three days [2], the reasons most commonly cited involve other holidays later in the month of Tishri.

[6] The decision to follow this path, which I considered when I first encountered the list of dechiyot, became easier when I learned from Bromberg's article that the same revision was proposed three centuries ago by the great mathematician K. F. Gauss.

If the first day of Rosh Hashanah were to fall on a Wednesday or Friday, then Yom Kippur, nine days later, would occur on a Friday or Sunday, and hence would be adjacent to Shabbat. This is undesirable for a number of reasons: (1) each of these days requires preparation, such as cooking, that cannot be performed on the other; (2) the dead cannot wait two days to be buried; (3) the shofar must be sounded in the evening after the end of Yom Kippur, but this act cannot be performed on Shabbat.

Furthermore, if Rosh Hashanah were to fall on a Sunday, then Hoshana Rabbah, the 21st day of Tishri, would occur on Shabbat. The observance of this holiday includes a procession of seven circuits around the synagogue's altar throughout which congregants vigorously wave palm leaves, ending with the beating of willow branches against the floor, symbolizing the elimination of sin. Anyone who has participated in this ritual will attest that this is work of the sort that is prohibited on Shabbat.

If the 2nd dechiyah were applied in isolation, then some years would have inadmissible lengths. The remaining two dechiyot are designed to circumvent this result. The 3rd addresses the case of a common year and the 4th the case of a leap year.

Third Dechihah: *If the delayed molad of a common year occurs on a Tuesday at or later than* $(15:204)$*, then Rosh Hashanah is delayed to the following Thursday.*

Fourth Dechihah: *If the delayed molad following a leap year occurs on a Monday at or later than* $(21:589)$*, then Rosh Hashanah is delayed to the next day.*

Note that at most one of the 2nd, 3rd, and 4th dechiyot can apply to any delayed molad. The 3rd advances Rosh Hashanah by two days, and each of the others by one. Furthermore, if the date of the molad is different from that of the delayed molad, then the latter must occur before midnight and the 3rd does not apply. Thus, Rosh Hashanah is delayed at most two days from the day of the true molad.

The motivation behind the 3rd and 4th dechiyot is easily explained. For example, in the case described in the 3rd, the days in which the delayed moladot of that year and the next would be 355 days apart. Consequently, the second molad falls on a Sunday and its Rosh Hashanah is postponed to Monday. If the first Rosh Hashanah were not delayed as well, then the length of the intervening year would be 366.

Such explanations, which are the only justifications to be found in the prior literature, fall short of a proof of the admissibility of every year. How do we know, for example, that a postponement of Rosh Hashanah according to the 3rd dechiyah does not adversely affect the length of the preceding year? Here we present a comprehensive proof in detail:

Let $\mathscr{M} = (d:h:p)$ and $\mathscr{M}' = (d':h':p')$ be the delayed moladot of years y and $y+1$, respectively. Our objective is to show that the length of y, as determined by $\mathscr{M}$, $\mathscr{M}'$, and the 2nd, 3rd, and 4th dechiyot, is one of the admissible values, 353, 354, 355, 383, 384, or 385.

Suppose first that y is a common year. Then

$$\mathscr{M}' = \mathscr{M} + 12 \cdot \mathscr{L} = \mathscr{M} + (354:8:876).$$

If $(h,p) < (15:204)$, then $d' = d + 354$ and the length of y is admissible unless the 3rd dechiyah applies to either $\mathscr{M}$ or $\mathscr{M}'$ and none of the three applies to the other. But the 3rd does not apply to $\mathscr{M}$, and if it applies to $\mathscr{M}'$, then d' is a Tuesday and since $354 = 50 \cdot 7 + 4$, d is a Friday and therefore subject to the 2nd.

We may assume, therefore, that $(h:p) \geq (15:204)$, which implies $d' = d + 355 = d + 50 \cdot 7 + 5$ and $(h':p') < (8:876)$. Thus, neither the 3rd nor the 4th applies to $\mathscr{M}'$. It follows that the length of y is admissible unless the 2nd applies to $\mathscr{M}'$ and none applies to $\mathscr{M}$. But if d' is a Wednesday or Friday, then d is a Friday or Sunday and the 2nd applies; if d' is a Sunday, the d is a Tuesday and the 3rd applies.

Now suppose y is a leap year. Then

$$\mathscr{M}' = \mathscr{M} + 13 \cdot \mathscr{L} = \mathscr{M} + (353:21:589).$$

If $(h:p) \geq (2:491)$, then $d' = d + 384$ and once again, the length of y is admissible unless the 3rd applies to either $\mathscr{M}$ or $\mathscr{M}'$ and none applies to the other. But since y is a leap year, the 3rd does not apply to $\mathscr{M}$. If it applies to $\mathscr{M}'$, then d' is a Tuesday, and since $384 = 54 \cdot 7 + 6$, d is a Wednesday and the 2nd applies to $\mathscr{M}$.

Thus, we may assume $(h:p) < (2:491)$, which implies $d' = d + 383 = d + 54 \cdot 7 + 5$ and $(h':p') \geq (21:589)$. Since neither the 3rd nor 4th applies to $\mathscr{M}$, the length of y is admissible unless the 2nd applies to $\mathscr{M}$ and none applies to $\mathscr{M}'$. But if d is a Friday or Sunday, then d' is a Wednesday or Friday and the 2nd applies to $\mathscr{M}'$, and if d is a Wednesday, then d' is a Monday and the 4th applies.

4 Keviyot

The length of a year and the day of the week on which it begins together determine the structure of the year, including the day of the week of every date. This pattern is known as the *keviyah* of the year. We have proved that every year is either defective (353 or 383 days), regular (354 or 384 days), or complete (355 or 385 days). It is also clear that Rosh Hashanah can occur only on a Monday, Tuesday, Thursday, or Shabbat. Rambam observed that of the $6 \times 4 = 24$ combinations of these parameters, only 14 keviyot are possible:

> If the New Year's Day of a year, whether ordinary [common] or embolismic [leap year], falls on a Tuesday, that year will always have regular months; if the New Year's day of a year, whether ordinary or embolismic, falls on a Saturday or Monday, it will never have regular months; if, however, it falls on a Thursday, we have to distinguish between the ordinary and the embolismic year; if it is an ordinary year, it can never have defective months, according to this system of calculation; if it is an embolismic year, it can never have regular months, according to this same system [9, Chapter VIII].

Of the 10 disallowed combinations, 9 are eliminated by the observation that Rosh Hashanah of $y+1$ cannot occur on Sunday, Wednesday, or Friday. The single remaining case is a year of length 385 that begins on a Tuesday. In this case, since 385 is divisible by 7, the following Rosh Hashanah must also be a Tuesday. Using the same notation as in the proof above, note that if the day d of $\mathscr{M}$ is the preceding Monday, then d' is either Shabbat $(d+383)$ or Sunday $(d+384)$, and according to the dechiyot, Rosh Hashanah of $y+1$ is Monday at the latest. Therefore, d must be Tuesday and d' is either Sunday $(d+383)$ or Monday $(d+384)$, and in the latter case, the time of $\mathscr{M}'$ must be earlier than $(21:589)$. Once again, in either case, Rosh Hashanah is Monday at the latest.

Much of the speculation surrounding the origin of the modern moladot is based on a correspondence between the sages of Babylonia and those of Palestine in the 10th century, pertaining to a controversy over the calculation of Rosh Hashanah of the year 4683. The letters, discovered in a Cairo synagogue around the end of the 19th century, refer to a meeting that had taken place some 80 years earlier, in which an agreement was apparently reached to base the molodot on Ptolemy's mean conjunctions. There is a consensus among historians [1, 12] that while this led the Babylonians to the establishment of Molad Adam as described in Section 2, the Palestinians instead took Nisan of the year 1 as their point of reference. The value derived for the molad for that month (by subtracting $37{,}266 \cdot \mathscr{L}$ from Ptolemy's conjunction of Nisan 3014) is 9 hours, 188 parts on a Wednesday, which was rounded to 9 hours. Since the Babylonians and Palestinians rounded up by 454 parts and down by 188 parts, respectively, the resulting moladot differed by 642 parts.

The first keviyah that was affected by this discrepancy was that of the year 4682 (921 C.E.). This was revealed when the Palestinian rabbis announced that Pesach of that year would begin on a Sunday, whereas the Babylonians had determined that it would begin the following Tuesday. The source of the disagreement was the 3rd dechiyah: the delayed molad of Tishri, 4683 derived from Molad Adam is $(1710089 : 15 : 441)$, while the corresponding Palestinian value was 642 parts earlier, $(1710089 : 14 : 879)$. Since this was on a Tuesday of a common year, the Babyloninans delayed Rosh Hashanah to Thursday, extending the year 4682 to a complete leap year of 385 days, while the Palestinians declared the year to be deficient. The controversy, which caused considerable distress throughout the Jewish world, was ultimately decided in favor of the Babylonians.

5 Mechanization of the Calendar

The computations discussed above are implemented by the program in the file `calendar.cpp`, which is coded in *RAC (Restricted Algorithmic C)*, a primitive subset of C++ designed for reasoning about computer hardware designs [10]. One feature of this language is its simplicity and readability, as it uses only the most basic C constructs. Another is its susceptibility to formal analysis—for this purpose, we have an automatic translator from RAC to the logical language of the ACL2 theorem prover [6], which enables formal verification of properties of RAC programs.

The program includes a procedure that computes the absolute date and time of day of the molad of a given year, as follows:

- The total number of preceding months, beginning with Tishri of year 1, is computed by summing the months of all previous years as determined by the Metonic cycle;
- This number is multiplied by the molad interval $\mathscr{L} = (29 : 12 : 793)$ and added to Molad Beharad, $\mathscr{B} = (2 : 5 : 204)$.

Two additional procedures convert between absolute and Hebrew dates. For example, the absolute date corresponding to a given Hebrew date (defined by month, day of month, and year) is derived as follows:

- The absolute date of Rosh Hashanah of the given year is computed from the molad according to the dechiyot;
- The length of the year is derived by similarly computing the absolute date of Rosh Hashanah of the following year and subtracting;
- The length of each month of the year is determined by the year length;
- The final result is derived by adding the absolute date of the day preceding Rosh Hashanah, the sum of the lengths of all preceding months of the year, and the day of the month.

The conversion between absolute and Gregorian dates is simpler, since every common year has 365 days and every leap year 366, independent of any other consideration. Thus, the absolute date corresponding to a given Gregorian date is easily computed once the Gregorian date of absolute day 1 is established as September 6, -3760. (Note that the year of this day is conventionally identified as 3761 B.C.E., since there is no Gregorian year 0.) Conversion between the Hebrew and Gregorian calendars is then performed by way of an intermediate computation of an absolute date. Conversions involving the Julian calendar are similarly handled.

In addition to mere automated computation, the translation of the program to ACL2 provides the capability of checking logical properties of the calendar mechanically using the heuristics of the theorem prover. These methods are commonly applied to the verification of correctness of computer designs in order to eliminate potentially costly errors. In the present case, we used the prover to verify several of the properties discussed in the preceding section. For the reader who is conversant in ACL2, the proof script resides in the file `proof.lisp`.

First, we formally prove that the length of every year belongs to the admissible set. In fact, the prover independently discovered the case analysis and worked through the details of the proof presented in Section 3. In view of the cumbersome nature of that proof, the high level confidence in correctness that this provides is welcome. As a consequence of this result, we prove Rambam's claim regarding the 20 realizable keviyot.

The same file also contains an ACL2 proof of Landau's claim that the 1st dechiyah ensures that the molad of every month occurs before the end of the first day of the month. The proof handles each admissible year length separately. For each month of the year, we compute the number of days elapsed from Tishri 1 to the first of the month (based on the lengths of months for the given year length) and the time elapsed between the molad of Tishri and that of the month (i.e., the appropriate multiple of $\mathscr{L}$). In the cases 355, 384, and 385, the claim follows for each month from the assumption that the molad of Tishri occurs before noon on Tishri 1. The same computation is inconclusive in the cases 353, 354, and 383, but in those cases, we succeed by using instead the corresponding constraint on the molad of the following year. For example, for a year of length 354, since the molad of the following year occurs before noon on Tishri 1 and the molad of the year of interest is earlier by an interval $12 \cdot \mathscr{L} = (354 : 8 : 876)$, it follows that the earlier molad occurs before $(7 : 204)$ of Tishri 1. This improved bound is sufficient to establish the claim.[7]

We conclude with some observations pertaining to the drift problems discussed in Section 1. Lunar drift is measured by the expression derived in Section 2 for the projected difference between the time of a molad and that of the corresponding lunar conjunction. According to this formula, a difference of approximately 15 days will have accumulated 33,000 years from now, by which time Rosh Chodesh will be celebrated under a full moon.

As for solar drift, the average length of the Hebrew year is $235/19 \cdot \mathscr{L} \approx 365.24683$ days, somewhat longer than the mean tropical year of 365.242199 days. Over the past 1000 years, this has resulted in an accumulated drift of 4.6 days. Continuing at this rate, we shall be celebrating "chag ha'aviv" in the fall in another 39,000 years.

It may seem excessive to worry about the plight of our descendants many millennia into the future, given that we are intent on leaving an uninhabitable planet to our grandchildren. But a more cheerful perspective is offered by rabbinical legal commentary:

> There is no need to worry, for certainly by that time the redemption will have occurred and we will go back to sanctifying the new moon through the testimony of witnesses. (Biur Halacha 427)

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